Anti-dissipative schemes for advection and application to Hamilton-Jacobi-Bellman equations

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Abstract: We propose two new anti-diffusive schemes for advection (or linear transport), one of them being a mixture of Roe's SuperBee scheme and of the UltraBee scheme. We show how to apply these schemes to treat time-dependant first order Hamilton-Jacobi-Bellman equations with discontinuous initial data, possibly infinitely-valued. Numerical tests are proposed, in one and two space dimensions, in order to validate the methods.

Key-words: Advection equation, antidiffusive scheme, ultrabee limiter, superbee limiter, Hamilton-Jacobi-Bellman equation.

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Schémas anti-diffusifs pour l'advection et application à des équations d'Hamilton-Jacobi-Bellman

Résumé : Nous proposons deux nouveaux schémas anti-diffusifs pour l'advection. L'un de ces schémas utilise une combinaison des limiteurs Superbee et Ultrabee. Nous montrons comment appliquer ces schémas pour résoudre des équations d'Hamilton-Jacobi-Bellman avec données intiales discontinues, éventuellement à valeurs infinies. Plusieurs tests numériques sont présentés pour comparer et valider les méthodes.

Mots-clés : Equation d'advection, schéma anti-diffusif, limiteur ultra bee, limiteur superbee, équation Hamilton-Jacobi-Bellman.

AMS subject classifications. Primary 65M12, Secondary 58J47.

1 Introduction

In this paper we first study two new schemes for the approximation of the linear first order advection equation with variable velocity $f(x)$:

$$\begin{aligned} v_t + f(x) \cdot v_x = 0 \qquad & x \in \mathbb{R},\ t > 0, \\ v(0, x) = \varphi(x) \qquad & x \in \mathbb{R}, \end{aligned} \tag{1}$$

on a uniform grid. The initial data $\varphi(x)$ may be discontinuous and infinitely valued, and the velocity may change signs.

The first scheme we propose is directly connected to the "Ultra-Bee" scheme of Roe [18]. This Ultra-Bee scheme has been recently reformulated by Després and Lagoutière [6, 14] in the context of non linear hyperbolic equations; while it is only of order one, it has the interesting property to tranport "exactly" a particular space of step functions in the case of linear advection $f(x) = const$. However in [6] only velocities $f(x)$ of constant sign are considered. Here we propose a simple "Ultra-Bee" generalisation to the case of changing sign velocities, still of order one.

Then we propose a second scheme (called "N-Bee"), that is basically of second order in the regions of regularity of the function and that has similar properties than the Ultra-Bee scheme where the function is discontinuous. This scheme can be seen as a mixture of Roe's "Super Bee" scheme [18] and of the Ultra-Bee scheme. It is also quite simple to implement.

Note that, in the context of scalar conservation laws, the Ultra-Bee scheme is not entropic. Entropic modifications are beeing developped (see F. Lagoutière [13] and [15], and F. Bouchut [4] for a second order entropic).

Secondly, we focus on the application, *without any theoretical proof* of these schemes to the discretization of non-linear first-order Hamilton-Jacobi-Bellman (HJB) equations of the following form:

$$\left\{ \begin{array}{l} v_t - \min\limits_{u \in U} \left(f(x,u) \cdot \nabla_x\, v + \ell(x,u) \right) = 0, \quad t > 0,\ x \in \mathbb{R}^n \\ v(0,x) = \varphi(x) \end{array} \right. \tag{2}$$

with $x \in \mathbb{R}^n$ ($n = 1$ or 2), $u \in U$ is a compact "admissible set" of $\mathbb{R}^p$, and the unkown $v = v(t, x)$ takes values in $]-\infty, +\infty]$. The initial data functions φ we consider are possibly discontinuous or infinitely-valued.

Such equations come from optimal control problems (v corresponds to the cost function of the optimized problem [2, 1], φ corresponds to a final cost), or from front-propagation problems (see [2, 19, 3]).

Several works deal with the numerical approximation of (2) in the case when the function φ is continuous. For instance, the Semi-Lagrangian schemes studied by Falcone, Ferretti et al. [7, 8] work well and are of high order in the regions of regularity.

However these schemes are diffusing and thus are not efficient for discontinuous φ. This remark holds for all schemes that use interpolation at some level, such as ENO [17] or WENO [12]. Note also that Discontinuous Galerkin method for Hamilton-Jacobi equations [11] (known to give very good accuracy) works only for continuous initial data.

We apply our schemes to some 1d and 2d HJB examples coming from optimal control problems and which are known in the literature. Even if we do not have theoretical convergence results in these case, the numerical results seem interesting and promising.

The paper is organized as follows. In section II, some preliminary definitions for the approximation of the non-conservative equation (1) are given. In section III, we define a first generalisation of the Ultra-Bee scheme to the case of changing sign velocities $f(x)$. In Section IV, we present the main scheme of the paper, hereafter refered as “N-Bee” scheme, for the approximation of (1). In Section V, we generalize the previous schemes to HJB equations (2) and give numerical results in one and two space dimensions.

2 Preliminaries

We look for a numerical approximation of the following advection equation with variable velocity $f(x)$:

$$\begin{cases} v_t + f(x)\, v_x = 0, & t > 0 \ x \in \mathbb{R} \\ v(0,x) = v_0(x) \end{cases} \tag{3}$$

We assume that $x \rightarrow f(x)$ is lipschitz-continous. The initial condition v_0 is assumed in $L^1_{loc}(\mathbb{R})$.

Let V_j^n denotes a numerical approximation to the solution $v(x_j, t_n)$, where $x_j = j\Delta x$, $t_n = n\Delta t$, and $\Delta x, \Delta t$ are the mesh sizes. We consider a numerical approximation of (3) in the following form:

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + f(x_j)\frac{V^{n,L}_{j+\frac{1}{2}} - V^{n,R}_{j-\frac{1}{2}}}{\Delta x} = 0, \tag{4}$$

and with the initialization:

$$V_j^0 := \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v_0(x)dx, \quad \text{where } x_{j+\frac{1}{2}} = (j+\frac{1}{2})\,\Delta x. \tag{5}$$

Here $V^{n,L}_{j+\frac{1}{2}}$ and $V^{n,R}_{j+\frac{1}{2}}$ are numerical *fluxes* that have to be defined. We write (4) in the equivalent non-conservative form:

$$V_j^{n+1} = V_j^n - \nu_j \left(V^{n,L}_{j+\frac{1}{2}} - V^{n,R}_{j-\frac{1}{2}}\right), \tag{6}$$

where

$$\nu_j := \frac{\Delta t}{\Delta x} f(x_j)$$

is the "local CFL" number. In the case $\nu_j = 0$, we thus consider $V_j^{n+1} = V_j^n$, and the fluxes $V_{j+\frac{1}{2}}^{n,R/L}$ need not to be defined. In the case $\nu_j \nu_{j+1} > 0$, these fluxes will always satisfy $V_{j+\frac{1}{2}}^{n,L} = V_{j+\frac{1}{2}}^{n,R}$. In particular, if $\nu_j \geq 0 \ \forall j$, then setting $V_{j+\frac{1}{2}}^n := V_{j+\frac{1}{2}}^{n,L}$, the scheme is simply written as:

$$V_j^{n+1} = V_j^n - \nu_j \left(V_{j+\frac{1}{2}}^n - V_{j-\frac{1}{2}}^n \right).$$

However, in the case $\nu_j \nu_{j+1} < 0$, i.e. when the speed $f(x)$ changes signs between x_j and x_{j+1}, we will have $V_{j+\frac{1}{2}}^{n,L} \neq V_{j+\frac{1}{2}}^{n,R}$ in general (see also Remark 2.1 below).

We consider the numerical approximation $V_{\Delta x, \Delta t}$ as a piecewise-constant function

$$V_{\Delta x,\Delta t}(t,x) = V_j^n, \qquad x_{j-1/2} < x < x_{j+1/2}, \qquad t_n \leq t < t_{n+1}. \tag{7}$$

For a real function $v \in L^1_{loc}(\mathbb{R})$, the total variation $TV(v)$ is defined in $[0, \infty]$ by

$$TV(v) := \sup \Big\{ \int_{\mathbb{R}} \phi' v, \ ||\phi||_\infty \leq 1, \ \phi \in C_0^1(\mathbb{R}) \Big\},$$

where $C_0^1(\mathbb{R})$ is the set of compact support C^1 functions. In particular, the total variation of $V^n = V_{\Delta x,\Delta t}(\cdot, t_n)$ is

$$TV(V^n) = \sum_j |V_{j+1}^n - V_j^n|. \tag{8}$$

Definition 2.1. *We say that the scheme is Total Variation Diminishing, or "TVD", if for all $n \geq 0$, $TV(V^{n+1}) \leq TV(V^n)$.*

We recall that a scheme written in Harten's incremental form [9, Prop. 3.6], i.e.,

$$V_j^{n+1} = V_j^n - C_{j-\frac{1}{2}}(V_j^n - V_{j-1}^n) + D_{j+\frac{1}{2}}(V_{j+1}^n - V_j^n), \qquad C_{j-\frac{1}{2}}, D_{j+\frac{1}{2}} \in \mathbb{R}, \tag{9}$$

is TVD if and only if the following conditions are realized for all j:

$$0 \leq C_{j+\frac{1}{2}}, \quad 0 \leq D_{j+\frac{1}{2}}, \quad \text{and} \quad C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1.$$

Note that a *linear* TVD scheme, even in the case $f(x) = c = const$, leads to an at most first-order scheme (see for instance Godlewski-Raviart [9, Chap. 3, Remark 3.2]). In order to have at least second order accuracy, we have to work with non-linear schemes.

Definition 2.2. *We say that a scheme (6) is L^∞-stable if the following holds*

$$\nu_j \geq 0 \ \Rightarrow \ \min(V_j^n, V_{j-1}^n) \leq V_j^{n+1} \leq \max(V_j^n, V_{j-1}^n), \tag{10a}$$

$$\nu_j \leq 0 \ \Rightarrow \ \min(V_j^n, V_{j+1}^n) \leq V_j^{n+1} \leq \max(V_j^n, V_{j+1}^n). \tag{10b}$$

Obviously, the above definition of stability implies the more usual L^∞-stability: $||V^{n+1}||_{L^\infty} \le ||V^n||_{L^\infty}$. Definition 2.2 also implies the TVD property in the case when $\nu_j \ge 0$, $\forall j$ [9]. This result is no more true if ν_j change signs, as we shall see in section 3.2.

Definition 2.3. *We say that a scheme (6) is consistent if all the fluxes $V^{n,L}_{j+\frac12}$ and $V^{n,R}_{j+\frac12}$ satisfy:*

$$\nu_j > 0 \;\Rightarrow\; \min(V^n_j, V^n_{j+1}) \le V^{n,L}_{j+\frac12} \le \max(V^n_j, V^n_{j+1}), \tag{11a}$$

$$\nu_{j+1} < 0 \;\Rightarrow\; \min(V^n_j, V^n_{j+1}) \le V^{n,R}_{j+\frac12} \le \max(V^n_j, V^n_{j+1}). \tag{11b}$$

Note that in the case when $\nu_j \ge 0$ for all j, the consistency consists in taking the flux $V^n_{j+\frac12} = V^{n,R}_{j+\frac12} = V^{n,L}_{j+\frac12}$ as a convex combination of V^n_j (stable scheme, but diffusive) and of V^n_{j+1} (antidiffusive scheme, but unstable).

Remark 2.1. *Let $j \in \mathbb{Z}$ such that $\nu_{j-1} > 0$, $\nu_j > 0$ and $\nu_{j+1} < 0$, $\nu_{j+2} < 0$. Then $V^{n,L}_{j-\frac12} = V^{n,R}_{j-\frac12}$ and $V^{n,L}_{j+\frac32} = V^{n,R}_{j+\frac32}$, but the values $V^{n,L}_{j+\frac12}$ and $V^{n,R}_{j+\frac12}$ are not necessarily equal for a consistent and stable scheme. For instance, if we assume that*

$$V^n_k = \begin{cases} 0 & \text{for } k \le j, \\ 1 & \text{for } k \ge j+1, \end{cases}$$

then the consistency condition implies $V^{n,R}_{j-\frac12} = V^{n,L}_{j-\frac12} = 0$, and (6) is equivalent to:

$$V^{n+1}_j = -\nu_j V^{n,L}_{j+\frac12}.$$

Now, the stability condition imposes $V^{n+1}_j \ge \min(V^n_j, V^n_{j-1}) = 0$ and thus $V^{n,L}_{j+\frac12} \le 0$. On the other hand, by the consistency condition, we have also

$$V^{n,L}_{j+\frac12} \ge \min(V^n_j, V^n_{j-1}) = 0,$$

hence $V^{n,L}_{j+\frac12} = 0$. By the same arguments, we obtain that $V^{n,R}_{j+\frac12} = 1$. We conclude that we have $V^{n,L}_{j+\frac12} \ne V^{n,R}_{j+\frac12}$.

3 Ultra-Bee scheme and generalizations

3.1 Ultra-Bee Scheme and Després-Lagoutière formulation

We assume in all this section that for all $x \in \mathbb{R}$,

$$f(x) > 0.$$

From now on, we drop the time index n and denote simply $V_j = V_j^n$ and $V_{j+\frac{1}{2}} = V_{j+\frac{1}{2}}^n$ when there is no ambiguity. Let

$$m_{j-1/2} := \min(V_j, V_{j-1}), \qquad M_{j-1/2} := \max(V_j, V_{j-1}), \tag{12}$$

and let the limiters b_j^+ and B_j^+ be defined by:

$$b_j^+ := M_{j-1/2} + \frac{1}{\nu_j}(V_j - M_{j-1/2}), \tag{13a}$$

$$B_j^+ := m_{j-1/2} + \frac{1}{\nu_j}(V_j - m_{j-1/2}). \tag{13b}$$

Under the CFL condition $0 < \nu_j \leq 1$, it is clear that the interval $[b_j^+, B_j^+]$ is non empty. The Ultra-Bee scheme, as formulated by Després and Lagoutière [6] (see also Lagoutière [14, chap. 1] for the non-conservative case), corresponds to the scheme (7) where the flux $V_{j+\frac{1}{2}}^{UB}$ is defined as the closest value to V_{j+1}^n in the interval $[b_j^+, B_j^+]$, i.e.,

$$V_{j+1/2}^{UB} := argmin\left\{|V - V_{j+1}^n|,\ b_j^+ \leq V \leq B_j^+\right\} = \min(\max(V_{j+1}^n, b_j^+), B_j^+). \tag{14}$$

It is proved in [6], under the CFL condition $0 < \nu_j \leq 1$, $\forall j$, that the UB scheme is consistent, L^∞-stable and TVD. The flux (14) can also be written as follows [5], using the formalism of Sweby [20]:

Lemma 3.1. *Assume that $0 < \nu_j \leq 1$, then the flux (14) is given by:*

$$V_{j+\frac{1}{2}}^{UB} := V_j + \frac{1-\nu_j}{2}\varphi_j(V_{j+1} - V_j), \tag{15}$$

where φ_j is defined, when $V_{j+1} \neq V_j$ and $\nu_j \neq 1$, by

$$\varphi_j = \max\left(0, \min\left(\frac{2r_j}{\nu_j}, \frac{2}{1-\nu_j}\right)\right), \quad \textit{where} \quad r_j = \frac{V_j - V_{j-1}}{V_{j+1} - V_j}, \tag{16}$$

and $\varphi_j = 0$ otherwise.

In the case of the linear advection with constant velocity $f(x) = c > 0$, the scheme advects "exactly" the cell-averages of a particular set of piecewise constant functions (or "step functions"), in the following sense. Assume the function (V_j^0) belongs to the set S defined by

$$S := \{(u_j),\ \exists \alpha \in [0,1],\ \forall j \in \mathbb{Z},\ u_{3j+1} = u_{3j},\ \text{and}\ u_{3j+2} = \alpha u_{3j+1} + (1-\alpha)u_{3j}\},$$

then by Theorem 3 of [6], under the CFL condition $0 < c\Delta t/\Delta x \leq 1$, the UB scheme satisfies for all j and $n \geq 0$:

$$V_j^n = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} v(t,x)dx. \tag{17}$$

The scheme keeps this exact advection property in two dimensions (or more) when using the Trotter spittling, in the case of advection of constant speed (i.e. $v_t + a\, v_x + b\, v_y = 0$, with a, b constant), as explained in [14, Chap 1] (see also [6]). Also, by [6, Th. 4], the UB scheme satisfies

$$||V_{\Delta t,\Delta x}(t_n,.) - v(t_n,.)||_{L^1(\mathbb{R})} \leq 3\Delta x\; TV(v_0). \tag{18}$$

This result indicates a convergence of order one, with a uniform bound in time. It has also an over-antidissipative (or "over-compressive") behavior as shown on various numerical examples [14, 6] (see the sinus example in Section 5.1).

3.2 An Ultra-Bee Generalized scheme

In this section, we propose a simple generalization of the Ultrabee scheme for changing signs velocities. (In Lagoutière [14] a formulation of UB scheme is proposed for negative velocities yet of constant sign.) This generalization, hereafter refered as "Ultra-Bee Generalized" and abreviated UB-G, is defined by (6) and the following:

- If $\nu_j > 0$ then define $V^{n,L}_{j+1/2} = \min(\max(V^n_{j+1}, b^+_j), B^+_j)$ as in (14).
- If $\nu_j < 0$ then define $V^{n,R}_{j-1/2}$ in a symmetric way as follows:

$$b^-_j := M_{j+1/2} + \frac{1}{|\nu_j|}(V_j - M_{j+1/2}),$$

$$B^-_j := m_{j+1/2} + \frac{1}{|\nu_j|}(V_j - m_{j+1/2}),$$

(recall that $m_{j+1/2} = \min(V^n_j, V^n_{j+1})$; $M_{j+1/2} = \max(V^n_j, V^n_{j+1})$), and

$$V^{n,R}_{j-1/2} := argmin\left\{|V - V^n_{j-1}|,\ b^-_j \leq V \leq B^-_j\right\} = \min(\max(V^n_{j-1}, b^-_j), B^-_j) \tag{19}$$

- If $\nu_j \leq 0$ and $\nu_{j+1} \geq 0$, then define

$$V^{n,R}_{j+\frac{1}{2}} := V_{j+1} \quad \text{and} \quad V^{n,L}_{j+\frac{1}{2}} := V_j. \tag{20}$$

- If $\nu_j \nu_{j+1} > 0$, then define $V^{n,R}_{j+\frac{1}{2}} := V^{n,L}_{j+\frac{1}{2}}$ (if $\nu_j > 0$) or $V^{n,L}_{j+\frac{1}{2}} := V^{n,R}_{j+\frac{1}{2}}$ (if $\nu_{j+1} < 0$).

Now let us check that the scheme is indeed well defined. We first remark that when $\nu_j \neq 0$, the flux $V^{n,L}_{j+\frac{1}{2}}$ is always defined. Indeed, if $\nu_j > 0$, then $V^{n,L}_{j+\frac{1}{2}}$ is defined by (14). If $\nu_j < 0$, then either $\nu_{j+1} < 0$ and $V^{n,L}_{j+\frac{1}{2}} = V^{n,R}_{j+\frac{1}{2}}$ defined as in (19), or $\nu_{j+1} \geq 0$ and $V^{n,L}_{j+\frac{1}{2}} = V_j$ as in (20).

On the other hand, if $\nu_j = 0$, then $V^{n,L}_{j+\frac{1}{2}}$ may not be defined in the case $\nu_{j+1} < 0$. However in this situation, $V^{n,L}_{j+\frac{1}{2}}$ is not needed in the scheme (6) because it only appears in the definition of V^{n+1}_j whose value will be set to V^n_j since $\nu_j = 0$. In the same way we can see that $V^{n,R}_{j+\frac{1}{2}}$ is always defined when $\nu_{j+1} \neq 0$, and does not need to be defined in the case $\nu_{j+1} = 0$.

Remark 3.1. *As in Lemma 3.1 we have an equivalent formulation of the UB-G scheme. Let* $\varphi^{UB}(r,\nu) := \max(0, \min(\frac{2r}{\nu}, \frac{2}{1-\nu}))$. *Then*

- *If* $\nu_j > 0$, $V^{n,L}_{j+\frac12} = V_j + \frac{1-\nu_j}{2}\varphi^{UB}(r_j,\nu_j)(V_{j+1}-V_j)$ *where* $r_j = \frac{V_j - V_{j-1}}{V_{j+1}-V_j}$;
- *If* $\nu_j < 0$, $V^{n,R}_{j-\frac12} := V_j + \frac{1-|\nu_j|}{2}\varphi^{UB}(r_j^-, |\nu_j|)(V_j - V_{j+1})$, *where* $r_j^- := \frac{V_{j+1}-V_j}{V_j - V_{j-1}} = \frac{1}{r_j}$;
- *If* $\nu_j \le 0$ *and* $\nu_{j+1} \ge 0$, *then* $V^{n,R}_{j+\frac12} := V_{j+1}$ *and* $V^{n,L}_{j+\frac12} := V_j$;
- *If* $\nu_j \nu_{j+1} > 0$, *then* $V^{n,R}_{j+\frac12} := V^{n,L}_{j+\frac12}$ *(if* $\nu_j > 0$*) or* $V^{n,L}_{j+\frac12} := V^{n,R}_{j+\frac12}$ *(if* $\nu_{j+1} < 0$*).*

Remark 3.2. *When* $\nu_j \le 0$ *and* $\nu_{j+1} \ge 0$, *the choice in (20) corresponds to the downwind (antidiffusive) flux in all cases, with no limiters.*

We first state a simple result.

Proposition 3.1. *Under the CFL condition* $|\nu_j| \le 1$, $\forall j$,
(i) the UB-G scheme is consistent,
(ii) the UB-G scheme is L^∞*-stable.*

Proof. (i) If $\nu_j > 0$ and $\nu_{j+1} > 0$ then we have $V^L_{j+\frac12} \in [m_{j+\frac12}, M_{j+\frac12}]$ by consistency of the UB scheme, and also $V^R_{j+\frac12} = V^L_{j+\frac12}$ by definition of the UB-G scheme. The case $\nu_j < 0$ and $\nu_{j+1} < 0$ is similar. In the case $\nu_j \le 0$ and $\nu_{j+1} \ge 0$, we have the consistency by definition (20). It remains to study the case $\nu_j \ge 0$ and $\nu_{j+1} \le 0$. We have either $\nu_j > 0$ and $V^L_{j+\frac12} \in [m_{j+\frac12}, M_{j+\frac12}]$ by construction, or $\nu_j = 0$ and in this case no condition is required on $V^L_{j+\frac12}$. In the same way, either $\nu_{j+1} < 0$ and $V^R_{j+\frac12} \in [m_{j+\frac12}, M_{j+\frac12}]$ by construction, or $\nu_{j+1} = 0$ and in this case no condition is required on $V^R_{j+\frac12}$.

(ii) If $\nu_j > 0$, since we have $V^L_{j+\frac12} \in [b_j^+, B_j^+]$ and $V^R_{j-\frac12} \in [m_{j-\frac12}, M_{j-\frac12}]$, we obtain $V_j^{n+1} \in [m_{j-\frac12}, M_{j-\frac12}]$. For $\nu_j < 0$, the arguments are the same. Finally, if $\nu_j = 0$ then $V_j^{n+1} = V_j^n$. Hence we obtain the L^∞ stability in all cases. ∎

Remark 3.3. *In the case of constant sign velocities, in [6] the* L^∞ *stability is obtained by assuming the limiters for the fluxes for all* j. *In the above proposition, when* $\nu_j \le 0$ *and* $\nu_{j+1} \ge 0$, *no limiters for* $V^R_{j+\frac12}$ *and* $V^L_{j+\frac12}$ *are needed for the* L^∞ *stability result, the consistency condition alone is sufficient. In fact, in this case, we could have taken for* $V^R_{j+\frac12}$ *and* $V^L_{j+\frac12}$ *any value between* V_j *and* V_{j+1} *in order to have an* L^∞*-stable scheme.*

Now we turn on studing the total variation of V^n. We shall say that x^* is a critical point of f if $f(x^*) = 0$.

Hypothesis (H) *All critical points* x^* *of* f *are such that either* f *is of constant sign in a neighbordhood of* x^*, *or* f *change signs at* x^*, *i.e., there exists* $\epsilon > 0$ *s.t. (a)* $\left[f_{|[x^*-\epsilon,x^*[} > 0,\ \text{and}\ \ f_{|]x^*,x^*+\epsilon]} < 0\right]$ *or (b)* $\left[f_{|[x^*-\epsilon,x^*[} < 0,\ \text{and}\ \ f_{|]x^*,x^*+\epsilon]} > 0\right]$. *Furthermore only a finite number of cases (b) may occur.*

Proposition 3.2. *Assume V^0 is such that $TV(V^0) < \infty$, f is lipschitz continuous and satisfies (H), and the CFL condition $|\nu_j| \le 1$. Then the UB-G scheme is of bounded total variation in the following sense: there exists a constant $C \ge 0$ (independant of the mesh and of V^0) such that $\forall n \ge 0$,*

$$TV(V^n) \le TV(V^0)(1 + C\Delta t). \tag{21}$$

Remark 3.4. *The technical assumption (H) is made in order that the number of index j such that $\nu_j < 0$ and $\nu_{j+1} > 0$ be bounded, since this will be the only place where the total variation of the scheme may increase.*

Proof. *(i) We first consider the case when $\nu_j < 0$ and $\nu_{j+1} > 0$ never happens simultaneously for Δx sufficiently small.* If $\nu_j \ge 0$, using the L^∞-stability property we have $V_j^{n+1} = V_j^n - C_{j-\frac{1}{2}}(V_j - V_{j-1})$ with $C_{j-\frac{1}{2}} \in [0,1]$, and we can write (9) with $D_{j+\frac{1}{2}} = 0$. In the case where $\nu_{j+1} \le 0$, we can also write V_{j+1}^{n+1} in the form (9) with $C_{j+\frac{1}{2}} = 0$. Hence in all cases we have the incremental form (9) with $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \le 1$ since one of the coefficient $C_{j+\frac{1}{2}}$ or $D_{j+\frac{1}{2}}$ always vanishes. We conclude that the scheme is TVD.

(ii) Now we consider the case when f change signs from negative to positive only once, i.e, for some $\epsilon > 0$ we have $f_{|[x^*-\epsilon,x^*[} < 0$, and $f_{|]x^*,x^*+\epsilon]} > 0$. More precisely, we suppose here $\nu_j < 0$ and $\nu_{j+1} > 0$ (with $\nu_{j-1} \le 0$ and $\nu_{j+2} \ge 0$ for Δx small enough). Because of the L^∞ stability, denoting $\Delta V_{j+\frac{1}{2}} := V_{j+1} - V_j$, we can write

$$\begin{aligned}
V_{j-1}^{n+1} &= V_{j-1}^n + D_{j-\frac{1}{2}}\Delta V_{j-\frac{1}{2}} && \text{(22a)}\\
V_{j}^{n+1} &= V_{j}^n + D_{j+\frac{1}{2}}\Delta V_{j+\frac{1}{2}} && \text{(22b)}\\
V_{j+1}^{n+1} &= V_{j+1}^n - C_{j+\frac{1}{2}}\Delta V_{j+\frac{1}{2}} && \text{(22c)}\\
V_{j+2}^{n+1} &= V_{j+2}^n - C_{j+\frac{3}{2}}\Delta V_{j+\frac{3}{2}} && \text{(22d)}
\end{aligned}$$

with coefficients C, D in $[0,1]$. We then obtain the local bound

$$\begin{aligned}
|\Delta V_{j-\frac{1}{2}}^{n+1}| + |\Delta V_{j+\frac{1}{2}}^{n+1}| + |\Delta V_{j+\frac{3}{2}}^{n+1}| \;\le\; & (1 - D_{j-\frac{1}{2}})|\Delta V_{j-\frac{1}{2}}^n| + (1 - C_{j+\frac{3}{2}})|\Delta V_{j+\frac{3}{2}}^n| \\
& + \left(|1 - C_{j+\frac{1}{2}} - D_{j+\frac{1}{2}}| + C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}}\right)|\Delta V_{j+\frac{1}{2}}^n|. \qquad (23)
\end{aligned}$$

In the case $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \le 1$ the scheme is TVD by Harten's criteria. Supposing that we are in the bad situation $TV(V^{n+1}) > TV(V^n)$ (otherwise there is nothing to prove), we thus have $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} > 1$ and either $C_{j+\frac{1}{2}} > 1/2$ or $D_{j+\frac{1}{2}} > 1/2$. We also obtain from (23) the following bound

$$TV(V^{n+1}) \le TV(V^n) + 2|\Delta V_{j+\frac{1}{2}}^n|. \tag{24}$$

There remains to bound $|\Delta V_{j+\frac{1}{2}}|$. By definition of V_{j+1}^{n+1}, using (22c), $V_{j+\frac{1}{2}}^R = V_{j+1}$, and $V_{j+\frac{3}{2}}^L = V_{j+1} + \frac{1-\nu_{j+1}}{2}\varphi_{j+1}(V_{j+2} - V_{j+1})$ where $\varphi_{j+1} = \varphi^{UB}(r_{j+1}, \nu_{j+1})$ by Remark 3.1.

We obtain

$$C_{j+\frac{1}{2}} \;=\; \nu_{j+1}\left(\frac{V^{n,L}_{j+\frac{3}{2}}-V^{n,R}_{j+\frac{1}{2}}}{V_{j+1}-V_j}\right)=\frac{1}{2}\nu_{j+1}(1-\nu_{j+1})\frac{\varphi_{j+1}}{r_{j+1}}. \tag{25}$$

Let us suppose that $C_{j+\frac{1}{2}} > 1/2$, the other case ($D_{j+\frac{1}{2}} > 1/2$) beeing similar. In this case we have $r_{j+1} = \frac{\Delta V_{j+\frac{1}{2}}}{\Delta V_{j+\frac{3}{2}}} \le 2\nu_{j+1}$, because otherwise, since $\varphi_{j+1} \le \frac{2}{1-\nu_{j+1}}$ by (16), we would have

$$C_{j+\frac{1}{2}} = \frac{1}{2}\nu_{j+1}(1-\nu_{j+1})\frac{\varphi_{j+1}}{r_{j+1}} \le \frac{\nu_{j+1}}{r_{j+1}} \le \frac{1}{2}.$$

In particular

$$|\Delta V_{j+\frac{1}{2}}| \le 2\nu_{j+1}|\Delta V_{j+\frac{3}{2}}| \le 2\nu_{j+1}\max_k |\Delta V^n_{k+\frac{1}{2}}|.$$

Now setting $M^n := |\max_k V^n_k - \min_\ell V^n_\ell|$, we have $|\Delta V^n_{k+\frac{1}{2}}| \le M^n$. Using the L^∞ stability, we see that $M^n \le M^0$. Also we have $M^0 \le TV(V^0)$. Hence we obtain $|\Delta V_{j+\frac{1}{2}}| \le 2\max(|\nu_j|,\nu_{j+1})TV(V^0)$. Furthermore there exists $x^* \in I_j = [x_j, x_{j+1}]$ such that $f(x^*) = 0$. So $|f(x)| \le L\Delta x$ for all $x \in I_j$, where L is a local lipschitz constant. Thus we have $\max(\nu_{j+1}, |\nu_j|) \le L\Delta t$.

At this point we have obtained $TV(V^{n+1}) \le TV(V^n) + C\Delta t TV(V^0)$ with $C = 4L$. However, the case $TV(V^{n+1}) > TV(V^n)$ in fact appears only in a particular configuration of the sequence $(V^n_{j-1}, V^n_j, V^n_{j+1}, V^n_{j+2})$ and we postpone to the appendix the proof of the following result:

Lemma 3.2. *Suppose that f changes signs only once, and let j be an index such that $\nu_j < 0$ and $\nu_{j+1} > 0$. If $TV(V^{n+1}) > TV(V^n)$ then $\forall m \ge n+1$ we have the TVD bound $TV(V^m) \le TV(V^{n+1})$.*

Hence if the scheme is not TVD at some time t^n, i.e. $TV(V^n) \le TV(V^{n-1}) \le TV(V^0)$ and $TV(V^{n+1}) > TV(V^n)$ then we still have for $k \ge n+1$, $TV(V^k) \le TV(V^{n+1}) \le TV(V^n) + C\Delta t TV(V^0) \le TV(V^0)(1 + C\Delta t)$, which is the desired bound.

(iii) Finally, in the case f changes signs $m \ge 1$ times from negative to positive: we obtain a similar bound with $C = 4mL$. Details are left to the reader. ■

Remark 3.5. *In the case $\nu_j < 0$, we have $C_{j+\frac{1}{2}} = \nu_{j+1}\big(\frac{V^{n,L}_{j+\frac{3}{2}}-V^{n,R}_{j+\frac{1}{2}}}{V_{j+1}-V_j}\big)$ (see (25)). Furthermore if $\nu_{j+1} > 0$ then $V^{n,L}_{j+\frac{3}{2}}$ is fixed, by definition of the UB-G scheme. Hence $V^R_{j+\frac{1}{2}} = V_{j+1}$ (resp. $V^L_{j+\frac{1}{2}} = V_j$) in (20) is the optimal choice of $V^R_{j+\frac{1}{2}}$ in the segment $[V_j; V_{j+1}]$ that minimizes $C_{j+\frac{1}{2}}$ (resp. $D_{j+\frac{1}{2}}$).*

Remark 3.6. *By adapting the standard arguments for scalar conservation laws of [9], we can show the convergence of the UB-G scheme to the weak solution of (3), i.e. the function $v \in L^1_{loc}((0,\infty)\times\mathbb{R})$ such that, for all $\varphi \in C^1(\mathbb{R}\times\mathbb{R})$,*

$$\int_{\mathbb{R}} v_0(x)\varphi(0,x)dx + \int_{(0,\infty)\times\mathbb{R}} \big\{v\varphi_t + v\big(f'\varphi + f\varphi_x\big)\big\}\, dt\ dx = 0.$$

4 The “N-Bee” scheme

The UB-G scheme presented in the previous section generalises simply the Ultra Bee scheme. Now we want to improve the order of the Ultra-Bee scheme. To do so, we use a formalism similar to the one of Sweby [20] for the non conservative equation (3).

4.1 Positive velocities

First we consider the case of positive velocities and assume the CFL condition

$$0 < \nu_j \leq 1.$$

We come back to the non-conservative form (6) with a flux defined by

$$V^R_{j+\frac{1}{2}} = V^L_{j+\frac{1}{2}} = V_{j+\frac{1}{2}} := V_j + \frac{1}{2}(1-\nu_j)\varphi_j(V_{j+1} - V_j), \quad \varphi_j \geq 0. \tag{26}$$

We look for a new function $\varphi_j = \varphi^{NB}(r_j, \nu_j)$ with $r_j = \frac{V_j - V_{j-1}}{V_{j+1} - V_j}$, such that the scheme be TVD, L^∞ stable, and of order 2. For this, we take a function φ^{NB} that satisfies $0 \leq \varphi^{NB} \leq \varphi^{UB}$, since this implies TVD and L^∞ stability, as proved in [5]. We also impose $\varphi^{NB}(1) = 1$ in order to have second order. We then choose the function defined by:

$$\varphi^{NB}(r, \nu) = \max(0, \min(1, \frac{2r}{\nu}), \min(r, \frac{2}{1-\nu})). \tag{27}$$

The function φ^{NB} is represented in Fig. 1.

We remark that for $\frac{1}{2} \leq r \leq 2$, we have $\varphi^{NB}(r) = \max(0, \min(1, 2r), \min(r, 2))$, as Roe's Super-Bee Scheme [20]. This case corresponds to smooth situations. For $r \leq \frac{\nu}{2}$ or $r \geq \frac{2}{1-\nu}$, we have $\varphi^{NB}(r, \nu) = \varphi^{UB}(r, \nu)$, as the Ultra-Bee scheme. This case corresponds to more rapid variations.

From now on, we call “N-Bee” the scheme corresponding to this function (27).

4.2 Changing sign velocities

In order to generalize the scheme to the case of changing sign velocities, we assume the CFL condition

$$|\nu_j| \leq 1, \tag{28}$$

and define the fluxes $V^{n,L}_{j+\frac{1}{2}}$ and $V^{n,R}_{j+\frac{1}{2}}$ by:

- $V^{n,L}_{j+\frac{1}{2}} := V_j + \frac{1}{2}(1-\nu_j)\varphi^{NB}(r_j, \nu_j)(V_{j+1} - V_j), \quad \text{if } \nu_j > 0$ (29a)
- $V^{n,R}_{j+\frac{1}{2}} := V_{j+1} + \frac{1}{2}(1-|\nu_{j+1}|)\varphi^{NB}(r^-_{j+1}, |\nu_{j+1}|)(V_j - V_{j+1}), \quad \text{if } \nu_{j+1} < 0$ (29b)
- $V^{n,L}_{j+\frac{1}{2}} = V^{n,R}_{j+\frac{1}{2}} := \frac{1}{2}(V_j + V_{j+1}) \quad \text{if } \nu_j \leq 0 \text{ and } \nu_{j+1} \geq 0$ (29c)
- $V^{n,L}_{j+\frac{1}{2}} = V^{n,R}_{j+\frac{1}{2}} \quad \text{if } \nu_j\nu_{j+1} > 0$ (29d)

with $r_j := \dfrac{V_j - V_{j-1}}{V_{j+1} - V_j}$ and $r^-_{j+1} := \dfrac{V_{j+1} - V_{j+2}}{V_j - V_{j+1}} = 1/r_{j+1}$. As for the UB-G scheme (see Section 3.2), we can see that the N-Bee scheme is well defined (we do not need to define $V^{n,L}_{j+\frac{1}{2}}$ in the case $\nu_j = 0$ or $V^{n,R}_{j+\frac{1}{2}}$ in the case $\nu_{j+1} = 0$).

We now give some elementary properties of this scheme

Proposition 4.1. *If $|\nu_j| \leq 1\ \forall j$, then the N-Bee scheme is consistent, L^∞-stable and TVD.*

Proof. The scheme is consistent by construction, and as in Prop. 3.1 (ii) the scheme is L^∞ stable. Then, as in the proof of Proposition 3.2, the main point is to check the TVD property in the case $\nu_j < 0$ and $\nu_{j+1} > 0$. It is clear that the N-Bee scheme can still be written in Harten's incremental form (9), and, as in (25), $C_{j+\frac{1}{2}} = \nu_{j+1}\big(\frac{V^{n,L}_{j+\frac{3}{2}} - V^{n,R}_{j+\frac{1}{2}}}{V_{j+1}-V_j}\big)$ and $D_{j+\frac{1}{2}} = -\nu_j\big(\frac{V^{n,L}_{j+\frac{1}{2}} - V^{n,R}_{j-\frac{1}{2}}}{V_{j+1}-V_j}\big)$. In order to obtain the TVD property we prove that $C_{j+\frac{1}{2}} \leq \frac{1}{2}$ and $D_{j+\frac{1}{2}} \leq \frac{1}{2}$. Taking into account the definition of $V^L_{j+\frac{3}{2}}$ by (29a) and $V^R_{j+\frac{1}{2}}$ as in (29c), and using that $\varphi_{j+1} := \varphi^{NB}(r_{j+1}, \nu_{j+1}) \leq \frac{r_{j+1}}{2\nu_{j+1}} \leq \frac{r_{j+1}}{\nu_{j+1}}$, we obtain

$$C_{j+\frac{1}{2}} = \nu_{j+1}\left(\frac{1}{2} + \frac{(1-\nu_{j+1})}{2}\frac{\varphi_{j+1}}{r_{j+1}}\right) \leq \nu_{j+1}\left(\frac{1}{2} + \frac{(1-\nu_{j+1})}{2}\frac{1}{\nu_{j+1}}\right) = \frac{1}{2}.$$

The proof of $D_{j+\frac{1}{2}} \leq \frac{1}{2}$ is similar. ∎

Remark 4.1. *As in Remark 3.6, the N-Bee scheme converges towards the weak solution of (3).*

Also, as in [9, pp. 189], we can show that the N-Bee scheme is of second order away from nonsonic extrema. More precisely, we have the following proposition (whose proof is postponed to the appendix):

Proposition 4.2. *Suppose that $(t,x) \in \mathbb{R}^*_+ \times \mathbb{R}$ is such that the solution v of (3) is locally smooth at (t,x), and that $v_x(t,x) \neq 0$. Then under the CFL condition and assuming that $\lambda = \frac{\Delta t}{\Delta x}$ is kept constant, the N-Bee scheme (29) is of order 2 at (t,x).*

4.3 A modified N-Bee scheme

The N-Bee scheme is of second order in the regular case but it is not as good as the Ultra-Bee scheme for the treatement of discontinuities. It is thus natural to combine both algorithms by choosing the N-Bee flux in regular regions and the Ultra-Bee flux in regions where a discontinuity is suspected. Concretely, the scheme is the following. We consider a given parameter $\delta > 0$.

If $|V^n_{j+1} - V^n_{j-1}| < \delta$ and $\nu_j > 0$, we take the N-Bee definition for $V^L_{j+\frac{1}{2}}$;
If $|V^n_{j+1} - V^n_{j-1}| < \delta$ and $\nu_j < 0$, we take the N-Bee definition for $V^R_{j-\frac{1}{2}}$;

If $|V^n_{j+1} - V^n_{j-1}| \geq \delta$ and $\nu_j > 0$, we take the UB-G definition for $V^L_{j+\frac{1}{2}}$;
If $|V^n_{j+1} - V^n_{j-1}| \geq \delta$ and $\nu_j < 0$, we take the UB-G definition for $V^R_{j-\frac{1}{2}}$.

(If $\nu_j = 0$ we simply consider $V^{n+1}_j = V^n_j$ and no definition for $V^L_{j+\frac{1}{2}}$ and $V^R_{j-\frac{1}{2}}$ is needed.)

We then obtain a "modified N-Bee scheme" for which we can prove consistency, L^∞ stability and TV-bounded properties as for the UB-G Scheme.

The parameter $\delta \geq 0$ should corresponds to the minimal discontinuity jump value of the function $V(.,x)$ we expect to detect. Although this discontinuity detector is very rough it gives quite acceptable numerical results as shown on the examples with discontinuous initial data.

Note that if $\delta = 0$ then we obtain the UB-G scheme, while if we set $\delta = +\infty$ we obtain the N-Bee scheme.

5 Numerical results.

Here we present some numerical results in one and two space dimensions. In section 5.1, we first consider advection equations. In section 5.2 we study the application of our algorithms to the case of HJB equations.

5.1 Advection equations

Example 1: sinus (one dimension). We consider the advection equation with a regular initial data:

$$\begin{cases} v_t + v_x = 0 \qquad 0 \leq x \leq 1; \\ v(x,0) = sin(2\pi x); \end{cases} \tag{30}$$

and with periodic boundary conditions. We compare the UB-G and N-Bee schemes. All tests are done using a CFL of 0.31.

The UB-G scheme, which is the same as the classical Ultra-Bee on this example, tends to project the solution on a class of step functions. As UB-G, the N-Bee scheme does not show diffusion. It furthermore gives a good approximation of the solution far away from the extrema, and for "small times" (see Fig. 2). We numerically check in Table 1 that N-Bee is of second order for the "local error", i.e. the L^∞ norm calculated on the interval $[0, 0.2] \cup [0.3, 0.7] \cup [0.8, 1]$ at time $t = 1$. Otherwise, it is of first order.

In Fig. 2, we compare the two schemes at time $t = 1$ and time $t = 5$ for $P = 50$ spatial mesh points. We observe, as is already known, that the error of UB is not amplified for larger times (this property is a conjecture [6]). The error with N-Bee is sensibly greater at time $t = 5$ than at $t = 1$. For N-Bee, there is a tendency to over-antidiffuse at the extrema (see Fig. 2 and Fig. 3). Yet we see in Fig. 3, for $t = 5$ and $P = 100$, that the error is significantly smaller with the N-Bee scheme.

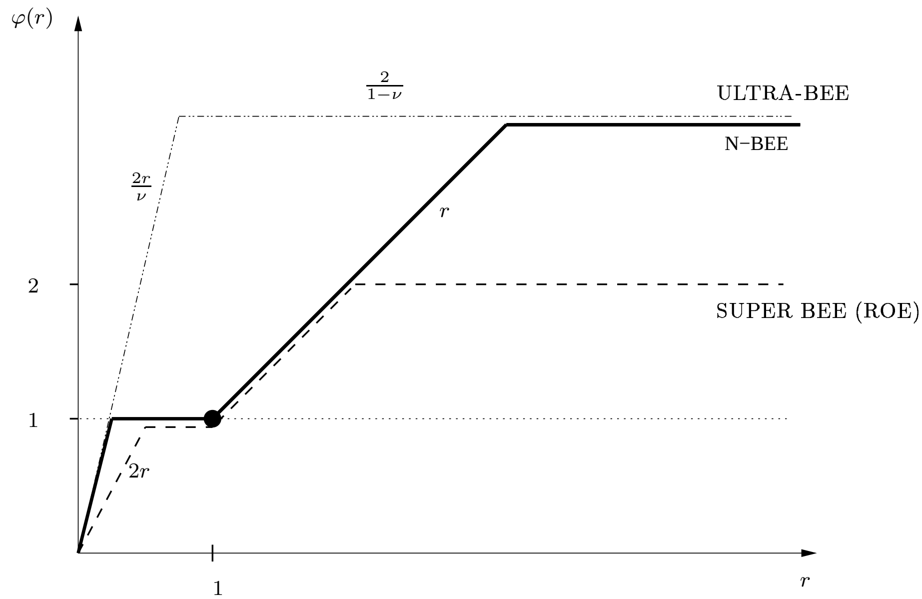
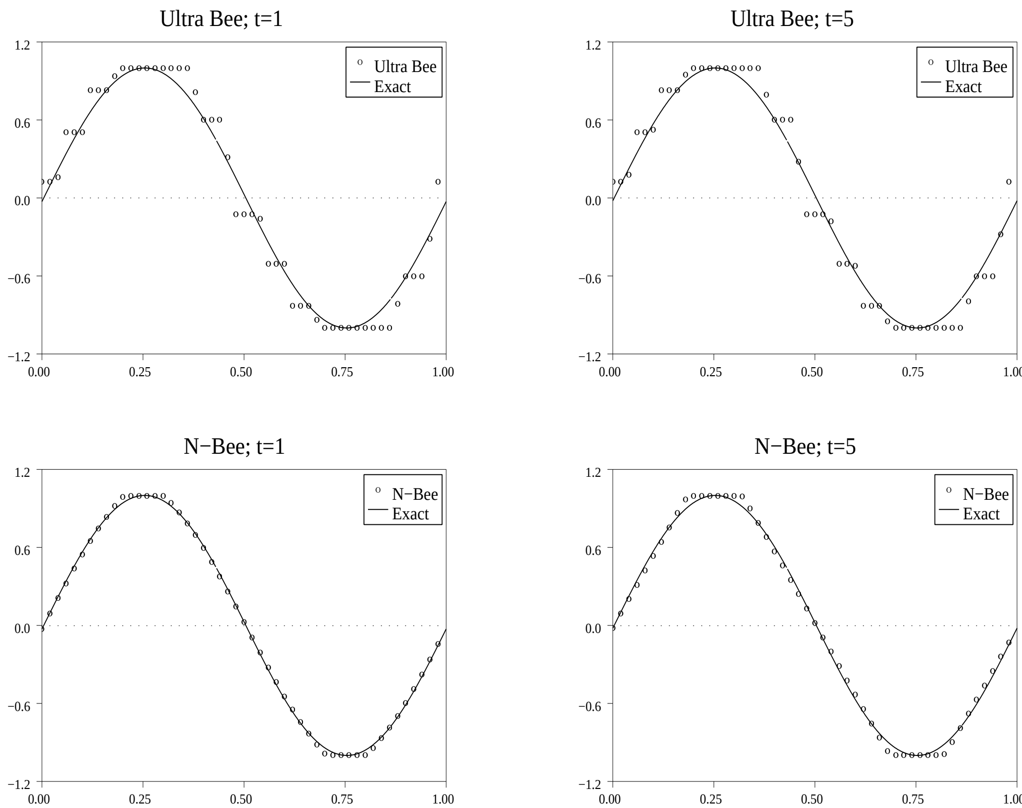


Figure 1: Function φ^{NB}

	Ultra bee				N-bee			
P	L^∞ error	L^1 error	loc. error	Order	L^∞ error	L^1 error	loc. error	Order
50	2.7E-01	8.4E-02	2.7E-01	-	4.5E-02	1.3E-02	4.5E-02	-
100	1.5E-01	4.0E-02	1.5E-01	0.83	1.8E-02	4.4E-03	9.0E-03	2.33
200	7.4E-02	1.8E-02	7.4E-02	1.06	8.3E-03	1.2E-03	1.4E-03	2.6
400	4.0E-02	8.5E-03	4.0E-02	0.88	3.3E-03	3.2E-04	3.8E-04	1.96
800	1.6E-02	3.6E-03	1.6E-02	1.32	1.4E-03	8.0E-05	9.7E-05	1.98

Table 1: Accuracy for example 1, $t = 1$



(a) (b)

Figure 2: Comparison between Ultra-Bee and N-Bee schemes (a) at $t = 1$, and (b) at $t = 5$, with $P = 50$ points

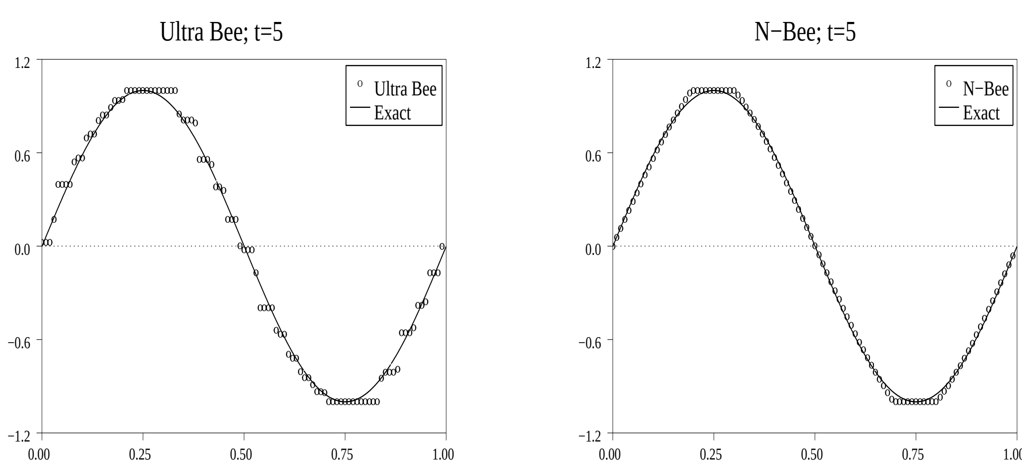


Figure 3: $P = 100$ points.

Example 2: Harten. Take the advection equation:

$$\left\{ \begin{array}{ll} v_t + v_x = 0 & -1 \leq x \leq 1; \\ v(x,0) = v_0(x); & \end{array} \right. \tag{31}$$

with periodic conditions at $x = \pm 1$, and Harten's initial data [10]:

$$v_0(x+0.5) = \left\{ \begin{array}{ll} -x\sin(\frac{3}{2}\pi x^2), & -1 < x < -\frac{1}{3}, \\ |\sin(2\pi x)|, & |x| < \frac{1}{3}, \\ 2x - 1 - \sin(3\pi x)/6, & \frac{1}{3} < x < 1. \end{array} \right.$$

We compare in Table. 2 the errors and cpu time of the Ultra-Bee, N-Bee, and A-Entropy (Antidiffusive Entropy [4]) schemes, with a CFL of 0.8. Here we use the modified N-Bee scheme with $\delta = 0.7$ (see Section 4.3) which corresponds to the minimal discontinuity we want to detect by the scheme.

We observe that the cpu time of N-Bee is roughly 1.4 times the cpu time of UB, and one-half of A-Entropy. The error in L^1 norm are very close for N-Bee and A-Entropy schemes.

Note that A-Entropy computes in fact *two values* per mesh interval, one which is an approximation of the average value, and the second an entropy value.

In Fig. 4 we compare N-Bee (with discontinuity detector) and A-Entropy, with $P = 100$ mesh points, at time $t = 1$.

In Fig. 5 we compare Ultra-Bee, N-Bee and A-Entropy schemes at longer time $t = 5$. We think the behavior of the N-Bee scheme is quite good for a relatively simple implementation.

Example 3: advection (2d) Now we consider the advection equation in 2 dimensions:

$$\left\{ \begin{array}{ll} v_t + \frac{1}{2}v_x + v_y = 0 & -1 \leq x, y \leq 1; \\ v(0,x,y) = v_0(x,y); & \end{array} \right. \tag{32}$$

with periodic boundary conditions at $x, y = \pm 1$, and with a (continuous) pyramidal initial data

$$v_0(x,y) := min(max(1-2|x|,0), max(1-2|y|,0)).$$

In this example (and the following 2d examples), the UB-G and N-Bee schemes are extended by using a Trotter splitting.

We compare in Fig. 6 the Ultra-Bee, N-Bee and exact solution first for a number of discretization mesh points $P_x = P_y = 20$ and then with $P_x = P_y = 40$. Computations are done with a CFL of 0.8. Note that here UB-G coincides with Ultra-Bee since the velocities do not change signs.

The solution is periodic of period $T = 4$, and the numerical solution is also computed at $t = 4$. On this example we see that N-Bee has a good behavior where the solution is non-constant, which is not the case for UB-G. Furthermore the two schemes (almost) do not diffuse outside the box $-1 \leq x, y \leq 1$.

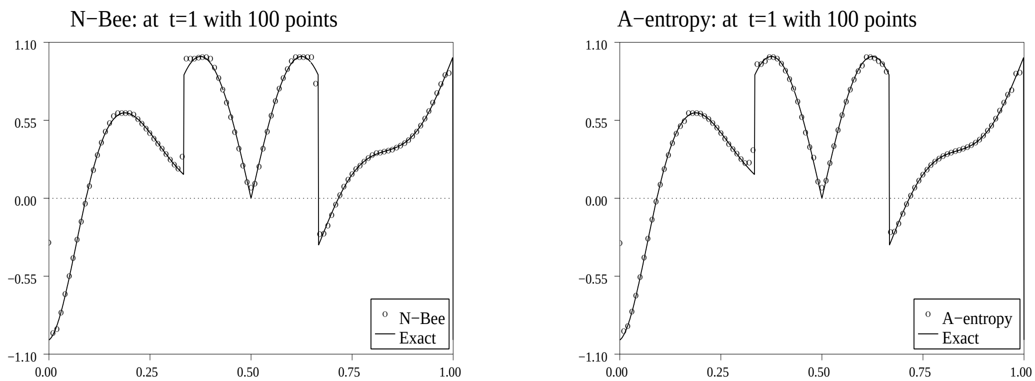


Figure 4: Comparison at time $t = 1$, $P = 100$ points, and $CFL = 0.8$.

	L^1 error			cpu time		
P	Ultrabee	N-bee	A-entropy	Ultrabee	N-bee	A-entropy
50	1.24E-01	4.61E-02	3.94E-02	0.11	0.16	0.21
100	5.87E-02	1.98E-02	1.61E-02	0.21	0.28	0.48
200	3.09E-02	1.01E-02	8.04E-03	0.44	0.59	1.16
400	1.43E-02	4.23E-03	3.93E-03	1.06	1.48	3.42
800	6.70E-03	1.74E-03	1.38E-03	3.01	4.27	11.22

Table 2: L^1 accuracy for example 2 (Harten's); $t = 1$, $CFL = 0.8$.

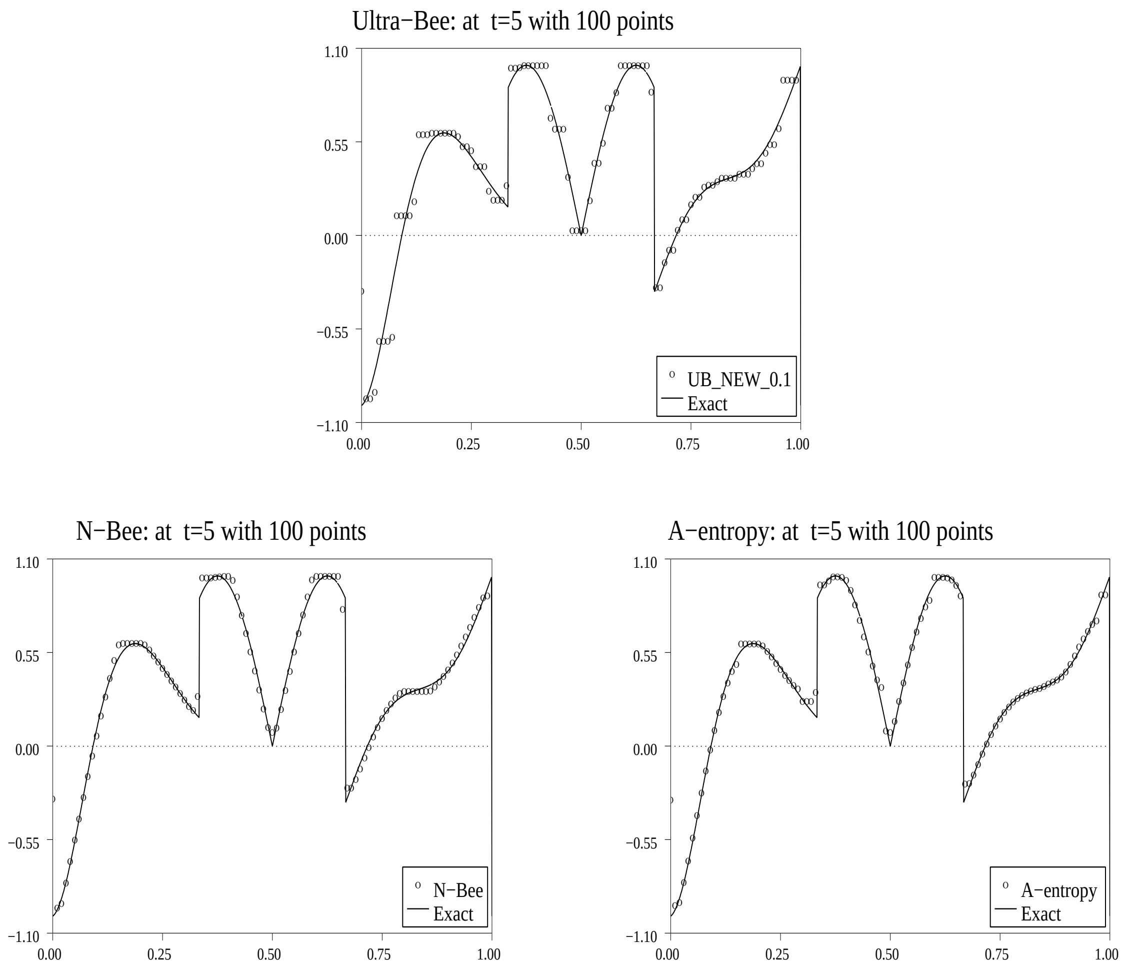
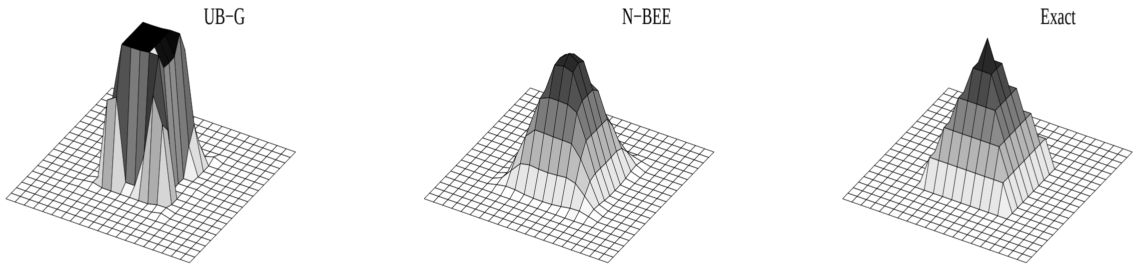
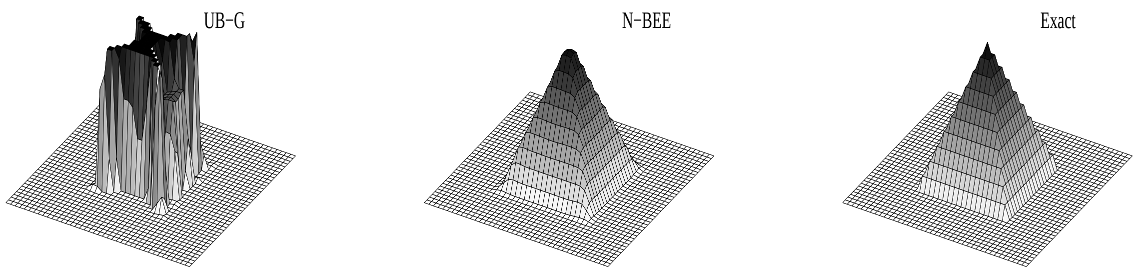


Figure 5: Ultra-bee, N-bee and A-entropy schemes on Harten's example.



(a) $P_x = P_y = 20$



(b) $P_x = P_y = 40$

Figure 6: Comparison of UB-G and N-Bee schemes.

In Fig. 7 we consider again equation (32) with trapezoidal initial data:

$$v_0(x,y) := \begin{cases} 1+x & \text{if } (x,y) \in [-0.5,0.5]^2 \\ 0 & \text{otherwise.} \end{cases}$$

We observe a good behavior of the (modified) N-Bee scheme.

Example 4: rotation (2d) We consider the following equation:

$$\begin{cases} v_t - y v_x + x v_y = 0 & -1.2 \le x, y \le 1.2; \\ v(0,x,y) = v_0(x,y); \end{cases} \tag{33}$$

with zero Dirichlet boundary conditions at $x, y = \pm 1.2$, and with a continuous initial data

$$v_0(x,y) := \frac{1}{R^2} \max(R^2 - (x-x_c)^2 - (y-y_c)^2, 0); \quad x_c = 0, \; y_c = 0.5, \; R = 0.5.$$

In this example, the velocities change signs.

We compare the UB-G scheme, the N-Bee scheme and the Exact solution in Fig. 8, at time $t = 2\pi$ which corresponds to a complete rotation of the initial data around the origin. The number of discretization mesh points is $P_x = P_y = 20$ and then $P_x = P_y = 40$. Computations are done with a CFL of 0.8. We still observe that there is almost no diffusion with the N-Bee outside the ball $|x - x_c|^2 + |y - y_c|^2 > R^2$, while there is a tendency to over-antidiffuse inside the ball (yet far less than the Ultra-Bee).

5.2 Application to HJB equations

Now we consider an extension of the UB-G scheme for the discretization of the HJB equation (2).

We first consider $(u_k)_{k=1,\dots,N_u}$ a given discretization of the admissible set U. Then the scheme is

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + \max_{u_k} \left(-f(x_j, u_k)[DV]_j^n(u_k) - \ell(x_j, u_k) \right) = 0$$

with

$$[DV]_j^n(u) := \frac{V_{j+\frac{1}{2}}^{n,L}(u) - V_{j-\frac{1}{2}}^{n,R}(u)}{\Delta x}$$

where $V_{j+\frac{1}{2}}^{n,L}(u)$ and $V_{j-\frac{1}{2}}^{n,R}(u)$ are the fluxes associated to the UB-G scheme for the discretization of $v_t - f(x,u)\, v_x = 0$ for a given $u \in U$, as defined in Sec. 3.2.

An extension of the N-Bee scheme for (3) is obtained in the same way.

Example 1 Let us consider the "target" problem

$$\begin{cases} v_t + \max_{u \in [-1,1]} \left(-\frac{1}{2}(u - x) \cdot v_x \right) = 0 \\ v(x,0) = \begin{cases} 0 & \text{if } x \in \mathcal{T} := [-c, c] \\ 1 & \text{otherwise} \end{cases} \end{cases} \tag{34}$$

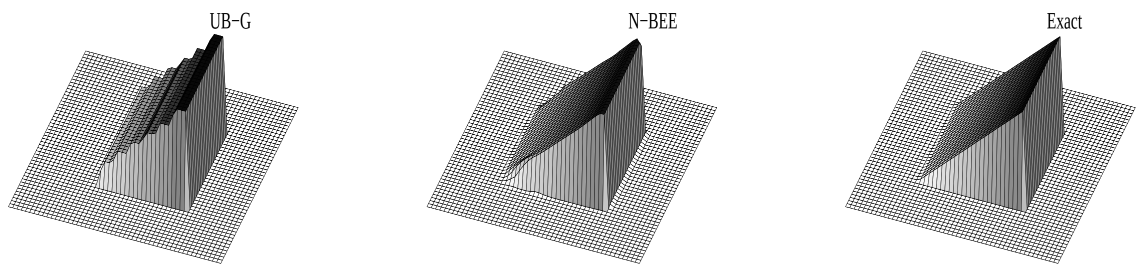
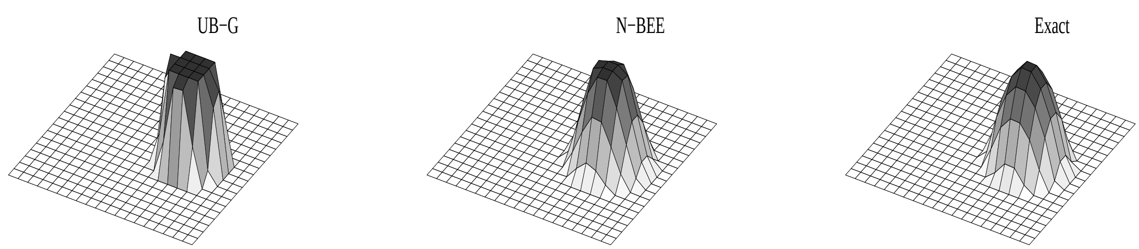
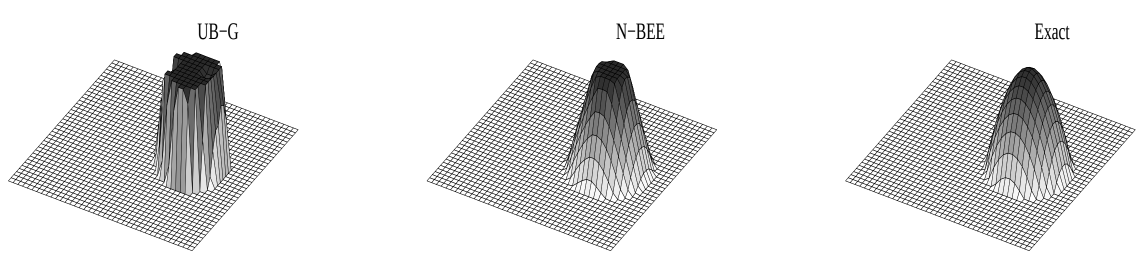


Figure 7: UB-G and modified N-Bee schemes (with $\delta = 0.4$) at $t = 4$; $P_x = P_y = 50$.



(a) $P_x = P_y = 20$



(b) $P_x = P_y = 40$

Figure 8: UB-G, N-Bee and exact solution at $t = 2\pi$.

with $c \geq 0$. (The first equation is the same as $v_t + \frac{1}{2}|v_x| - \frac{x}{2}v_x = 0$.) This problem comes from an Optimal Control problem [16, 3, 1]. The 0-value set of the function $v(.,t)$ corresponds to the region of points x from which the target $\mathcal{T}$ can be reached at time t by trajectories of $\dot{y}(s) = \frac{1}{2}(u(s) - y(s)))$ and $y(0) = x$, with $u(s) \in [-1, 1]$. The exact (viscosity) solution of this problem reads:

$$v(x,t) = \begin{cases} 0 & \text{if } x \in [1 - (c+1)e^{t/2}, -1 + (c+1)e^{t/2}]; \\ 1 & \text{if not.} \end{cases}$$

We compare our method (using the UB-G scheme) with the Semi-Lagrangian method which is known to give good results for regular functions. Here, for the Semi-Lagrangian scheme, we have used an explicit 4-th order Runge-Kutta method in order to compute the characteristics.

Figure 9 shows the numerical solution computed for $t = 1.5$ with $c = 0.5$ (large target).

Figure 10 shows the numerical solution for a thin target, with $c = \frac{\Delta x}{2}$. This is an approximation of the case $c = 0$ (this gives a computed initial data $v_j^0 = 0$ if $j \neq 0$ and $v_j^0 = 1$ if $j = 0$).

For these tests, we have taken $N_u = 3$ and $\{(u_k)\} = \{-1, 0, 1\}$. We see here in Fig. 9 and 10 that there is no diffusion using the Ultra-Bee (or UB-G) scheme, to the contrary to a Semi-Lagrangian scheme.

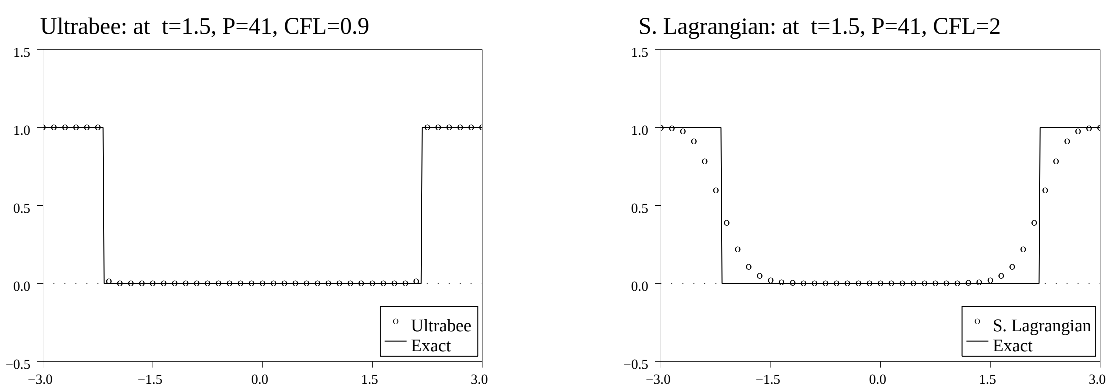


Figure 9: Ultra-bee and Semi-Lagrangian schemes, $\mathcal{T} = [-0.5, 0.5]$.

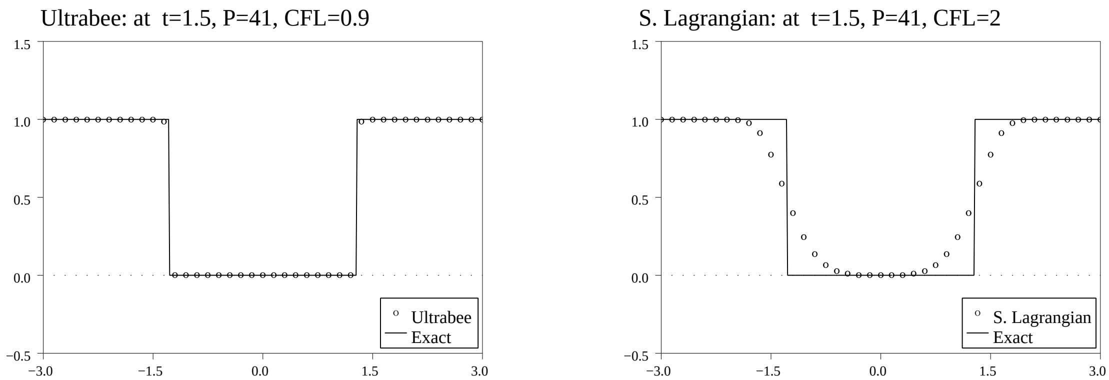


Figure 10: Ultra-bee and Semi-Lagrangian schemes, $\mathcal{T} = [-\frac{\Delta x}{2}, \frac{\Delta x}{2}]$.

Example 2 We consider a one dimensional "Burger's equation":

$$\begin{cases} v_t + \dfrac{(v_x+1)^2}{2} = 0 \quad -1 < x < 1 \\ v(x,0) = -\cos(\pi x) \end{cases} \tag{35}$$

computed as $v_t + \max_{u \in [-\pi+1, \pi+1]} \left(-u(v_x+1) + \dfrac{u^2}{2} \right) = 0$. We have used a CFL of 0.5 and an approximation with $N_u = 20$ control mesh points.

Results are shown in Fig. 11 and in Table 3, where the L^∞ error is only computed in the region $[-1, -0.95] \cup [-0.75, 1]$ (where the solution is regular).

On this example N-Bee is less precise than high-order methods such as ENO schemes of Osher and Shu (see Table in [17]). However we still observe a good numerical convergence.

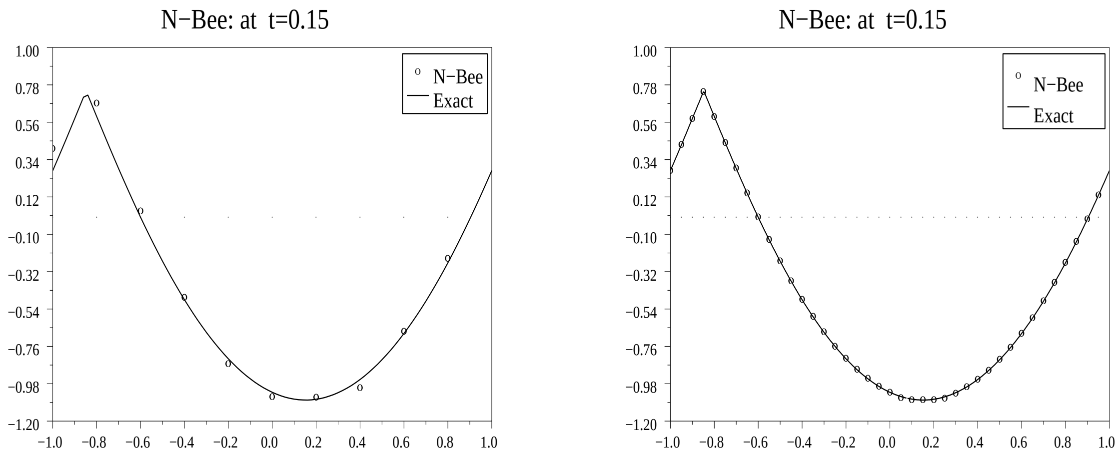
Remark 5.1. *We have also tested the UB-G scheme but we have observed that there is no convergence towards the solution.*

Example 3 (Expanding holes): We consider the following HJB equation:

$$\begin{cases} v_t + f(x,y) \cdot \nabla v + ||\nabla v|| = 0, \qquad -3 < x, y < 3; \\ v(0,x,y) = v_0(x,y), \qquad -3 < x, y < 3. \end{cases} \tag{36}$$

with $f(x,y) = (-y, x)$, zero boundary conditions at $x, y = \pm 3$ (and $|| \cdot ||$ denotes the Euclidean norm). The initial data corresponds to two "separate holes" centered in $A = (-1, 0.5)$ and $B = (1, -0.5)$:

$$v_0(x,y) = \begin{cases} 0 & \text{if } (x+1)^2 + (y-0.5)^2 \leq 0.25 \text{ or } (x-1)^2 + (y+0.5)^2 \leq 0.25 \\ 1 & \text{otherwise} \end{cases}$$



(a) 10 points (b) 40 points

Figure 11: One-dimensional equation, “Burger”, $t = 1.5/\pi^2$.

The approximation is done as follows:

$$v_t + \max_k (f(x,y) \cdot \nabla v + u_k \cdot \nabla v) = 0.$$

The exact solution of the continuous problem is used for reference (see [17] and references therein).

We compare in Fig. 12 the Ultra-Bee scheme, the N-Bee scheme and the exact solution with $P_x = P_y = 30$ mesh points. Computations are done with a CFL of 0.967. We remark that the fronts are well localized within a one-mesh size. The meeting of the two fronts at $t = 0.5$ (coming from each expanding hole), and then the merging of the fronts, at $t = 1$, is also well reproduced, with no diffusion problems.

Example 4 (bell): For this last example, we consider the HJB equation:

$$\begin{cases} v_t + \frac{1}{2}||\nabla v||^2 = 0, & -1 < x, y < 1; \\ v(0,x,y) = v_0(x,y), & -1 < x, y < 1. \end{cases} \tag{37}$$

with zero boundary conditions at $x, y = \pm 1$, and with a initial data corresponding to a “bell”:

$$v_0(x,y) = max(0, 1 - ||(x,y)||^2)$$

In the exact solution appears a singularity at time $t = 0.5$. We compare in Fig. 13 the N-Bee scheme and the exact solution with $P_x = P_y = 19$ mesh points at time $t = 0.4$ (before singularity) and $t = 0.5$. Computations are done with a CFL of 0.9. The numerical resolution is done using the equivalent formulation of (37):

$$v_t + \max_{||u|| \le 1} (u \cdot \nabla v + \tfrac{1}{2}||u||^2) = 0,$$

with $N_u = 37$ control values located inside the unit ball. The exact solution for the continuous problem is known and used for reference.

	$t = 0.5/\pi^2$		$t = 1.5/\pi^2$	
No of points	L^1 error	L^∞ error	L^1 error	L^∞ error
$P = 10$	2.9E-02	2.73E-02	6.73E-02	1.31E-01
$P = 20$	1.15E-02	1.22E-02	1.43E-02	2.18E-02
$P = 40$	3.95E-03	8.17E-03	5.04E-03	8.55E-03
$P = 80$	1.11E-03	3.78E-03	2.18E-03	2.76E-03

Table 3: L^1 error and L^∞ error in smooth regions for Example 2.

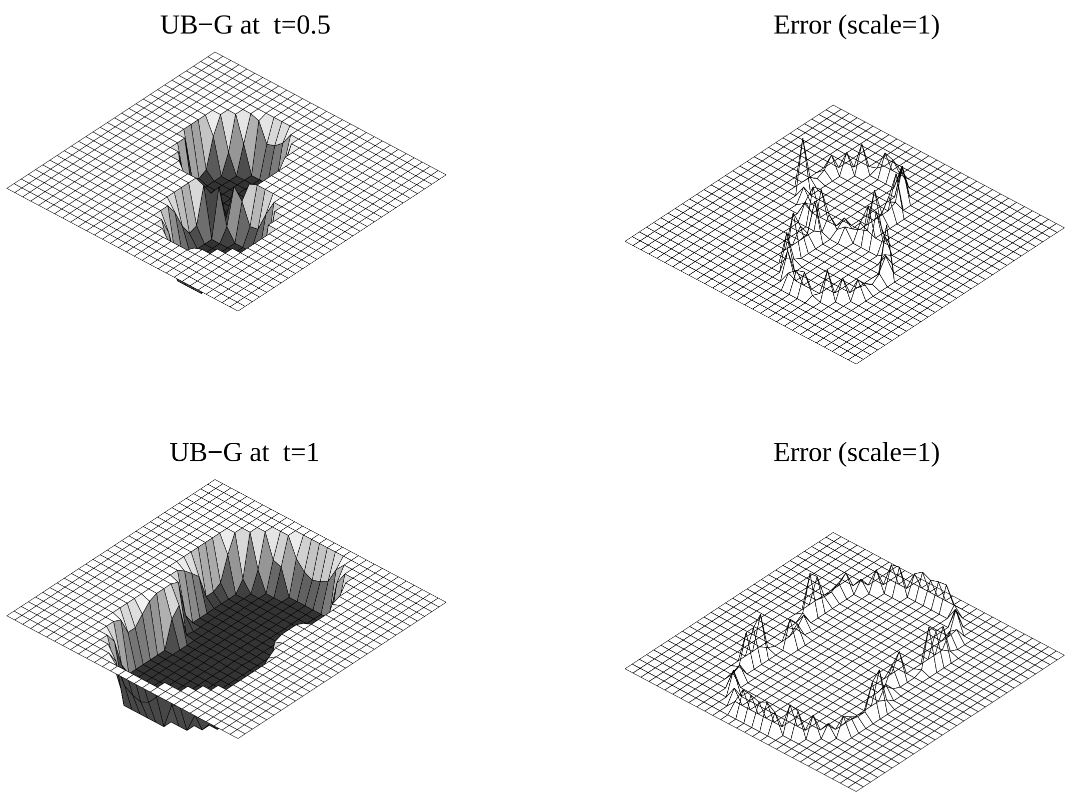


Figure 12: UB-G and error at time $t = 0.5$ and $t = 1$.

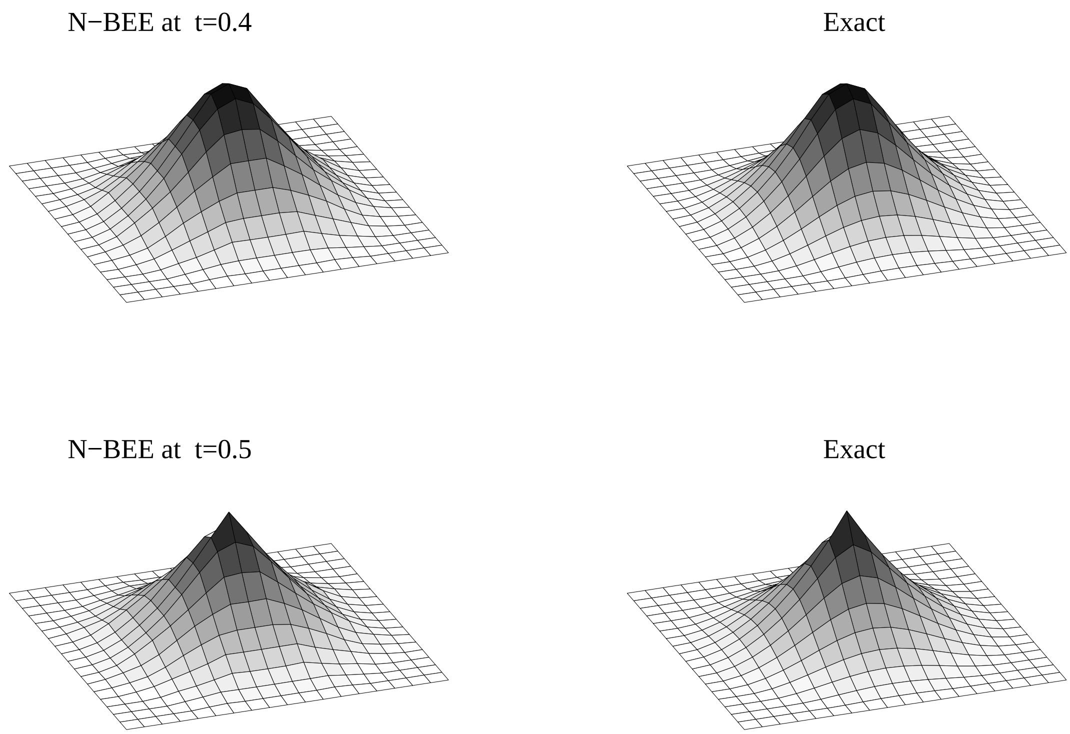


Figure 13: N-Bee and exact solution at time $t = 0.4$ and $t = 0.5$.

A Proof of Lemma 3.2

We recall that we are in the situation where f changes signs only once and there exists an index j such that $\nu_j < 0$ (and $\nu_{j-1} \leq 0$) and $\nu_{j+1} > 0$ (and $\nu_{j+2} \geq 0$). We divide the proof into two steps, given by Lemma A.1 and Lemma A.2 below. In this case we have seen that UB-G can be written in the form $V_k^{n+1} = V_k^n - C_{k-\frac{1}{2}}(V_k - V_{k-1}) + D_{k+\frac{1}{2}}(V_{k+1} - V_k)$ with $C_{k+\frac{1}{2}} + D_{k+\frac{1}{2}} \leq 1$ forall $k \neq j$. In the case we have also $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1$ then we can conclude that the scheme is TVD.

Lemma A.1. *We assume that the "bad" situation $TV(V^{n+1}) > TV(V^n)$ occurs, with f vanishing only once, and that there exists an index j such that $\nu_j < 0$ and $\nu_{j+1} > 0$. Then we have*
(i) $(V_{j-1}^n, V_j^n, V_{j+1}^n, V_{j+2}^n)$ is a monotone sequence,
(ii) $(V_{j-1}^{n+1}, V_j^{n+1}, V_{j+1}^{n+1}, V_{j+2}^{n+1})$ is a non-monotone sequence.

Proof. Step (i) Let us suppose that $(V_{j-1}^n, V_j^n, V_{j+1}^n, V_{j+2}^n)$ is not a monotone sequence, and show that $TV(V^{n+1}) \leq TV(V^n)$ in this case.

If (V_{j-1}, V_j, V_{j+1}) *is monotone:* we can suppose for instance that $V_{j-1} \leq V_j \leq V_{j+1}$. In this case, $V_{j+2} < V_{j+1}$. and thus $V_{j+1} \geq max(V_j, V_{j+2})$, and $\varphi^{UB}(r_{j+1}, \nu_{j+1}) = 0$. Recalling that we are treating the case when $\nu_{j+1} > 0$ and $\nu_j < 0$, we thus have $V_{j+\frac{3}{2}}^L = V_{j+1}$ and $V_{j+\frac{1}{2}}^R = V_{j+1}$ by Remark 3.1. Using (25) we obtain $C_{j+\frac{1}{2}} = 0$. Hence $C_{j+\frac{1}{2}} + D_{j+\frac{1}{2}} \leq 1$, which implies, by the TVD criteria, that $TV(V^{n+1}) \leq TV(V^n)$. (In the case $V_{j-1} \geq V_j \geq V_{j+1}$, we have $V_{j+1} < V_{j+2}$ and we also obtain $TV(V^{n+1}) \leq TV(V^n)$ in the same manier.)

If (V_{j-1}, V_j, V_{j+1}) *is non-monotone:* we see that either $V_j \geq \max(V_{j-1}, V_{j+1})$ or $V_j \leq min(V_{j-1}, V_{j+1})$. In both cases we obtain $\varphi^{UB}(r_j^-, |\nu_j|) = 0$. Then, by Remark 3.1 (and recalling that $\nu_j < 0$) we obtain $D_{j+\frac{1}{2}} = 0$, and thus $TV(V^{n+1}) \leq TV(V^n)$. This concludes the proof of (i).

Step (ii) When the bad situation $TV(V^{n+1}) \geq TV(V^n)$ occurs, then by step (i) the sequence $S^n := (V_{j-1}^n, V_j^n, V_{j+1}^n, V_{j+2}^n)$ is monotone. We can suppose without loss of generality that the sequence is non-decreasing, i.e., $V_{j-1}^n \leq V_j^n \leq V_{j+1}^n \leq V_{j+2}^n$. First we note that $\gamma_{j+1} := \frac{(1-\nu_{j+1})}{2}\varphi^{UB}(r_{j+1}, \nu_{j+1}) \geq 0$. Recalling that $V_{j+\frac{1}{2}}^{n,R} = V_{j+1}$ and using Remark 3.1 we obtain

$$\begin{aligned} V_{j+1}^{n+1} &= V_{j+1}^n - \nu_{j+1}(V_{j+\frac{3}{2}}^{n,L} - V_{j+\frac{1}{2}}^{n,R}) \\ &= V_{j+1}^n - \nu_{j+1}\gamma_{j+1}(V_{j+2}^n - V_{j+1}^n) \leq V_{j+1}^n. \end{aligned} \tag{38}$$

In the same way we can show that

$$V_j^{n+1} \geq V_j^n. \tag{39}$$

Let us prove furthermore the following

$$V_{j-1}^{n+1} \leq \min(V_j^{n+1}, V_{j+1}^{n+1}) \leq \max(V_j^{n+1}, V_{j+1}^{n+1}) \leq V_{j+2}^{n+1}. \tag{40}$$

We shall only check the left side inequality, i.e. $V_{j-1}^{n+1} \leq \min(V_j^{n+1}, V_{j+1}^{n+1})$, the other side beeing similar. Since $\nu_{j-1} < 0$, using the L^∞ stability of the scheme and (39) we have

$$V_{j-1}^{n+1} \leq \max(V_{j-1}^n, V_j^n) = V_j^n \leq V_j^{n+1}.$$

On the other hand, using that $\nu_{j+1} \geq 0$, the L^∞ stability of the scheme, and (38) we have $V_{j-1}^{n+1} \leq V_j^n = \min(V_j^n, V_{j+1}^n) \leq V_{j+1}^{n+1}$. Hence (40) is proved.

Now, let us show that $V_j^{n+1} > V_{j+1}^{n+1}$, which will prove that the sequence S^{n+1}, in view of (40), is non-monotone. On the contrary, suppose that $V_j^{n+1} \leq V_{j+1}^{n+1}$, and define

$$\begin{aligned} \Sigma_{n+1} &:= |\Delta V_{j-\frac{1}{2}}^{n+1}| + |\Delta V_{j+\frac{1}{2}}^{n+1}| + |\Delta V_{j+\frac{3}{2}}^{n+1}| \\ &= |V_j^{n+1} - V_{j-1}^{n+1}| + |V_{j+1}^{n+1} - V_j^{n+1}| + |V_{j+2}^{n+1} - V_{j+1}^{n+1}| \end{aligned}$$

(with the notation $\Delta V_{j+\frac{1}{2}} = V_{j+1} - V_j$). Then we have

$$\begin{aligned} \Sigma_{n+1} = V_{j+2}^{n+1} - V_{j-1}^{n+1} &= V_{j+2}^n - C_{j+\frac{3}{2}}\Delta V_{j+\frac{3}{2}} - (V_{j-1}^n + D_{j-\frac{1}{2}}\Delta V_{j-\frac{1}{2}}) \\ &= \Delta V_{j+\frac{3}{2}}\,(1 - C_{j+\frac{3}{2}}) + V_{j+1}^n - (V_j^n - (1 - D_{j-\frac{1}{2}})\Delta V_{j-\frac{1}{2}}) \\ &= |\Delta V_{j+\frac{3}{2}}|(1 - C_{j+\frac{3}{2}}) + |\Delta V_{j+\frac{1}{2}}| + |\Delta V_{j-\frac{1}{2}}|(1 - D_{j-\frac{1}{2}}). \end{aligned}$$

By straightforward calculations (using Harten's incremental form (9)), we get

$$TV(V^{n+1}) \leq \sum_{k<j} |\Delta V_{k+\frac{1}{2}}|(1 - C_{k+\frac{1}{2}}) + |\Delta V_{j+\frac{1}{2}}| + \sum_{k>j+1} |\Delta V_{k+\frac{1}{2}}|(1 - D_{k+\frac{1}{2}}).$$

Hence $TV(V^{n+1}) \leq TV(V^n)$, and this is contradictory with the assumption of the Lemma. We have thus proved assertion (ii). ■

Lemma A.2. *We consider the same assumptions as Lemma A.2. Then we have* $\forall k \geq n+1$, $(V_{j-1}^k, V_j^k, V_{j+1}^k, V_{j+2}^k)$ *non-monotone, and the total variation is bounded by* $TV(V^k) \leq TV(V^{n+1})$.

Proof. By the previous Lemma, S^n is monotone. Let us suppose that S^n is non-decreasing. As in the proof of the previous Lemma we obtain

$$(H_{n+1}): \qquad V_{j-1}^{n+1} \leq V_j^{n+1}, \quad V_j^{n+1} > V_{j+1}^{n+1}, \quad V_{j+1}^{n+1} \leq V_{j+2}^{n+1}.$$

By definition of the UB-G scheme, $V_{j+\frac{1}{2}}^{n+1,L} = V_j^{n+1}$ (since $\nu_j \leq 0$ and $\nu_{j+1} \geq 0$), and also $\varphi^{UB}(r_j^-, |\nu_j|) = 0$ and $V_{j-\frac{1}{2}}^{n+1,R} = V_j^{n+1}$. Hence we obtain $V_j^{n+2} = V_j^{n+1}$. By the same argument we get also that $V_{j+1}^{n+2} = V_{j+1}^{n+1}$.

On the other hand we have $\nu_j \leq 0$ and $\nu_{j+1} \geq 0$. Thus $V_{j-1}^{n+2} \leq \max(V_{j-1}^{n+1}, V_j^{n+1}) = V_j^{n+1}$. In the same way, $V_{j+2}^{n+2} \geq V_{j+2}^{n+1}$. We conclude then, for $k = n+2$:

$$(H_k): \qquad V_{j-1}^k \leq V_j^k, \quad V_j^k > V_{j+1}^k, \quad V_{j+1}^k \leq V_{j+2}^k.$$

We can also prove recursively that H_k holds for $k \geq n+2$. Then the proof of $TV(V^k) \leq TV(V^{n+1})$ for $k \geq n+1$ is similar to the proof of (ii) of Lemma A.1. ■

B Proof of Proposition 4.2.

For sequences such that $\Delta t \to 0$, $t_n = n\Delta t \to t$ and $x_j = j\Delta x \to x$, and assuming that $\frac{\Delta t}{\Delta x} = \lambda$ where $\lambda > 0$ is kept constant, we want to prove that the consistency error ϵ_j^n is of order $O(\Delta x^2)$, where

$$\epsilon_j^n := \frac{v(t_{n+1}, x_j) - v(t_n, x_j)}{\Delta t} + f(x_j)\frac{v^L_{j+\frac{1}{2}} - v^R_{j-\frac{1}{2}}}{\Delta x}. \tag{41}$$

In (41) we have denoted $v^L_{j+\frac{1}{2}}$ and $v^R_{j-\frac{1}{2}}$ the estimation of the flux from the exact solution $\bar{V}_j^n = v(t_n, x_j)$. We treat here only the case where $f(x) > 0$, for which we have $\nu_{j-1} > 0$ and $\nu_j > 0$ for Δx small enough. The proof of the other cases, where $f(x) < 0$ or $f(x) = 0$, is left to the reader. Hence we have $v^L_{j+\frac{1}{2}} = \bar{V}_j + \frac{1}{2}(1-\nu_j)\varphi_j(\bar{V}_{j+1} - \bar{V}_j)$ and $v^R_{j-\frac{1}{2}} = \bar{V}_{j-1} + \frac{1}{2}(1-\nu_{j-1})\varphi_{j-1}(\bar{V}_j - \bar{V}_{j-1})$ where $\varphi_k := \varphi^{NB}(r_k, \nu_k)$ for $k = j, j-1$ as in (27), and $r_k = \frac{\bar{V}_k - \bar{V}_{k-1}}{\bar{V}_{k+1} - \bar{V}_k}$. We write

$$\epsilon_j^n \;=\; \frac{1}{\Delta t}\left(\bar{V}_j^{n+1} - \bar{V}_j^n + \nu_j(\bar{V}_j^n - \bar{V}_{j-1}^n) + \frac{\nu_j}{2}\delta\right), \tag{42}$$

where

$$\delta := \varphi_j(1-\nu_j)(\bar{V}_{j+1}^n - \bar{V}_j^n) - \varphi_{j-1}(1-\nu_{j-1})(\bar{V}_j^n - \bar{V}_{j-1}^n).$$

With the notations $v' = v_x(t_n, x_j)$, $v'' = v_{xx}(t_n, x_j)$ and using Taylor expansions we have

$$\bar{V}_{j+1}^n - \bar{V}_j^n = \Delta x v' + \frac{\Delta x^2}{2} v'' + O(\Delta x^3) \tag{43}$$

$$\bar{V}_j^n - \bar{V}_{j-1}^n = \Delta x v' - \frac{\Delta x^2}{2} v'' + O(\Delta x^3) \tag{44}$$

$$\bar{V}_{j-1}^n - \bar{V}_{j-2}^n = \Delta x v' - \frac{3\Delta x^2}{2} v'' + O(\Delta x^3), \tag{45}$$

and thus

$$r_j = 1 + \Delta x \frac{v''}{v'} + O(\Delta x^2) \quad \text{and} \quad r_{j-1} = 1 + \Delta x \frac{v''}{v'} + O(\Delta x^2). \tag{46}$$

Also for $x \in V(0^\pm)$, we have $\varphi^{NB}(1+x) = 1 + a^\pm x$ (with $a^+ = 1$ and $a^- = 0$). If $\epsilon = \pm 1$ denotes the sign of $\frac{v''}{v'}$, using (46) we obtain

$$\varphi_j = \varphi^{NB}(r_j, \nu_j) = 1 + a^\epsilon \Delta x \; v' \; v'' + O(\Delta x^2) \tag{47}$$

and also the same estimate for φ_{j-1}.

Now, using (43)-(45) we have

$$\delta = \Big((1-\nu_j)\varphi_j - (1-\nu_{j-1})\varphi_{j-1}\Big)\; \Delta x \; v' + \frac{1}{2}((1-\nu_j)\varphi_j + (1-\nu_{j-1})\varphi_{j-1}) \; \Delta x^2 \; v'' + O(\Delta x^3) \tag{48}$$

By (47) and since $1-\nu_{j-1} = 1-\nu_j + O(\Delta x)$, we obtain the estimate

$$\frac{1}{2}((1-\nu_j)\varphi_j + (1-\nu_{j-1})\varphi_{j-1})\Delta x^2\ v'' = (1-\nu_j)\Delta x^2\ v'' + O(\Delta x^3). \tag{49}$$

By Taylor expansions and using $\Delta t = O(\Delta x)$ we have $\nu_{j-1}-\nu_j = -\Delta t f'(x_j) + O(\Delta x^2)$, and together with (47) we obtain the following estimate

$$\begin{aligned}(1-\nu_j)\varphi_j - (1-\nu_{j-1})\varphi_{j-1} &= (1-\nu_j)(\varphi_j - \varphi_{j-1}) + (\nu_{j-1}-\nu_j)\varphi_{j-1} \\ &= -\Delta t f'(x_j) + O(\Delta x^2)\end{aligned} \tag{50}$$

Taking (49) and (50) into (48) gives

$$\delta = -\Delta t \Delta x\ f'(x_j)\ v' + (1-\nu_j)\Delta x^2\ v'' + O(\Delta x^3) \tag{51}$$

Denoting $\dot v = v_t(t_n, x_j)$ and $\ddot v = v_{tt}(t_n, x_j)$ we also obtain by Taylor expansions

$$\bar V_j^{n+1} - \bar V_j^n + \nu_j(\bar V_j^n - \bar V_{j-1}^n) = \dot v\ \Delta t + \frac{1}{2}\ddot v\ \Delta t^2 + O(\Delta t^3) + \nu_j(\Delta x\ v' - \frac{1}{2}\Delta x^2\ v'' + O(\Delta x^3)) \tag{52}$$

Finally, using $\dot v = -f(x_j)v'$ and $\ddot v = f(x_j)f'(x_j)v' + f(x_j)^2 v''$ and recalling that $\nu_j = \frac{\Delta t}{\Delta x} f(x_j)$ and $\frac{\Delta t}{\Delta x} = const$ after some computations we obtain from (42), (51) and (52) the estimate $\epsilon_j^n = O(\Delta x^2)$ as desired. ■

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